MA 111, Topic 3: Compensation

Our next topic is the mathematics behind an important part of society: **Compensation!**

Definition 1 (Topic Idea: Compensation). Society must deal with a number of scenarios where two or more people have a legitimate claim on some desired resource(s).

Often in these situations the people involved cannot (or will not) agree on the value of the disputed resource(s).

Compensation is the process of awarding ownership of resource(s) and (typically) money to each of the people involved in the dispute.

Example 2 (Typical Compensation Scenarios). Here are some general scenarios where compensation is important:

- Inheritance
- Divorce Settlements
- Treaties/International Disputes

Compensation Basics

Definition 3 (Related Idea: Compensation Basics 1).

- **People** who have a claim on a resource. People can be described by name or by some other designation (like Person 1, 2, ...).
- N for the number of people involved in a given scenario.
- We use lower-case b, decorated with a subscript, to describe the **Value/Bid** that each person makes. If there is more than one resource then this represents the **total bid** for that person.

Example 4 (Compensation Basics 1). Kim and Kanye must share a bag of chips. In this scenario $N = 2$ because there are two people. If Kim (Person 1) thinks the chips are worth \$10 and Kanye (Person 2) thinks the chips are worth \$6 then

$$b_1 = 10 \quad \text{while} \quad b_2 = 6.$$

Example 5 (Caesar and Cleopatra 1). Caesar and Cleopatra are going through a rough break-up! Caesar thinks Cleo should give The Port of Alexandria to him, thinking the The Port of Alexandria is worth 200 thousand denarii. Cleopatra knows the it isn't worth that much and could easily get another one for only 160 thousand denarii.

- What is N in this scenario?

$N = 2$

- What is b_{Caesar}?

$b_{Caesar} = 200$

- What is b_{Cleo}?

$b_{Cleo} = 160$

Definition 6 (Related Idea: Compensation Basics 2).

- The person who bids the most for a resource is called the **highest bidder**. Sometimes we use the letter h to denote the **highest bid**. This is the value of the bid for the highest bidder.
- The person who wins the resource is called the **winning bidder**. We use the letter w to denote the **winning bid**. Depending on the compensation method we use, *THE WINNING BIDDER DOES NOT HAVE TO BE THE HIGHEST BIDDER!!!*

Example 7 (Compensation Basics 2). Recall in the previous example Kim thinks the chips are worth \$10 and Kanye thinks the chips are worth \$6. If Kanye is awarded the chips then

$$w = 6 \quad \text{while} \quad h = 10.$$

Example 8 (Mean Girls 1, Part 1). Three soon-to-be-former friends decide to never see one another again. Unfortunately they share a prized possession: **The Burn Book** . The chart below represents what they each would bid on this item:

	Regina	**Gretchen**	**Karen**
The Burn Book	\$300	\$210	\$240

Suppose in the end that Gretchen gets The Burn Book .

- What is N for this example?

$N = 3$

- What is h?

$h = 300$

- What is w?

$W = 210$

Fair Share

Definition 9 (Related Idea: Fair Share). When N people divide a resource, a person's **fair share** is the amount worth $\frac{1}{N}$ of the total value of the resource *in their eyes*.

Using our notation this can be calculated using each of the bids:

- The fair share for Person 1 is $\frac{b_1}{N}$.
- The fair share for Person 2 is $\frac{b_2}{N}$.
- The fair share for Person 3 is $\frac{b_3}{N}$.

$\vdots$

- The fair share for Person N is $\frac{b_N}{N}$.

Example 10 (Compensation Method: Random). Suppose two people place bids of $b_1 = 4000$ and $b_2 = 3500$ for an antique. Since $N = 2$ we calculate the *Fair Share of Person 1* to be $\frac{4000}{2} = 2000$ while the *Fair Share of Person 2* is $\frac{3500}{2} = 1750$.

Example 11 (Mean Girls 1, Part 2). Three soon-to-be-former friends decide to never see one another again. Unfortunately they share a prized possession: **The Burn Book** . The chart below represents what they each would bid on this item:

	Regina	**Gretchen**	**Karen**
The Burn Book	\$300	\$210	\$240

- What is the fair share for Regina ?

$\frac{300}{3} = 100$

- What is the fair share for Gretchen ?

$$\frac{219}{3} = 70$$

- What is the fair share for Karen ?

$$\frac{240}{3} = 80$$

Example 12 (Arrested Development 1). Lindsay and Tobias decide to divorce (again!). Since they don't own much there aren't a lot of shared assets to fight over. However, there are two items that both people want; **Diamond Skin Cream** and a **Poofy Blouse/Shirt**. Here are their bids:

	Lindsay	**Tobias**
Diamond Skin Cream	\$200	\$260
Poofy Blouse/Shirt	\$100	\$80

- What is N for this example?

$N = 2$

- What is the fair share for Lindsay ?

$$\frac{200+100}{2} = 150$$

- What is the fair share for Tobias ?

$$\frac{260+80}{2} = 170$$

Example 13 (Miley and Liam 1). Miley and Liam have broken up and must divide some items they bought together. The chart below represents what they each would bid on these items (all in thousands of dollars):

	Miley	Liam
Engagement Ring	\$400	\$250
Warehouse of Teddy Bear Costumes	\$100	\$60
Basic Human Dignity	\$30	\$200

- What is N for this example?

$N = 2$

- What is the fair share for Miley ?

$\frac{400+100+30}{2} = 265$

- What is the fair share for Liam ?

$\frac{250+60+200}{2} = 255$

Definition 14 (Related Idea: Compensation Basics 3).

- We need a way to write the **compensation** for people. We will exclusively use x, decorated with a subscript for *whose* compensation, to describe the overall value a person receives.

  So x_1 is the compensation that Person 1 receives, x_2 is the compensation that person 2 receives. We could also use names instead of numbers in the subscripts.

  These amounts are also sometimes called **payouts**.

Example 15 (Related Idea: Compensation Basics 3). Three sisters, April, May, and June, must decide who will inherit their parents' house. They make the following bids (in thousands of dollars):
$b_{April} = \$200 \qquad b_{May} = \$400 \qquad b_{June} = \$900$
The court awards the house to June who must pay April \$150 (thousand) and May \$350 (thousand). Then

$$x_{April} = \$150 \qquad x_{May} = \$350 \qquad x_{June} = \$400.$$

Example 16 (Crazy Good Tacos 1). The local taco truck operated by *Habeñero Loco* routinely loses meal orders. Three people whose orders were lost discuss how much they would like to be the next in line. Katy is really hungry and would pay \$9 extra to be next in line. Elizabeth and Cara are not as hungry and would pay \$6 and \$3, respectively, to be next in line.

Suppose that Cara is the next person to get her order.

- What are x_{Cara} and her Fair Share?

  $X_{Cara} = 3$, fair share is $\frac{3}{3} = 1$

- What are x_{Katy} and her Fair Share?

  $X_{Katy} = 0$, fair share is $\frac{9}{3} = 3$

- What are $x_{Elizabeth}$ and her Fair Share?

  $X_{Elizabeth} = 0$, fair share is $\frac{6}{3} = 2$

- How do Katy and Elizabeth feel about this compensation arrangement?

  Since neither of them are compensated they probably do not like the deal

Compensation Method: Knaster's Procedure

Definition 17 (Compensation Method: Knaster's Procedure).

1(a) All N people secretly bid on a resource.

1(b) The highest bidder receives the resource but pays w for the item.

1(c) Everyone (including the highest bidder) receives their fair share. In symbols, everyone gets $\frac{b_i}{N}$ at the end of stage 1.

Leftover money is called the **surplus**. This is the excess paid to the 3rd-party mediator and will generally be *negative*.

2 Everyone (including the winner) receives an equal portion of the surplus. This amount paid back to each person is *positive*.
Everyone gets extra money in addition to what was received above in Step 1.

Compensation=Fair Share + Surplus Portion

Knaster's method is also called **The Method of Sealed Bids**.
The steps of Knaster's Procedure can be described in very simple ways:

Step 1 Everyone gets their fair share. People who get item(s) must pay so they only get their fair share.

Step 2 Everyone gets an equal portion of the surplus.

Compensation=Fair Share + Surplus Portion

We can easily set up Knaster's Compensation to reflect these ideas:

	List of N People	
Fair Share	*Info you fill in*	
Value of Item(s) Received	*Info you fill in*	Total ↓
Pays/Gets	**Fair Share -Value of Items**	**=Surplus**
Compensation=	**Fair Share** $+\frac{1}{N}\cdot$ **Surplus**	

Example 18 (Crazy Good Tacos 2)**.** Recall the bids to be next in line at the *Habeñero Loco* taco truck:

	Katy	**Elizabeth**	**Cara**
Next in Line	\$9	\$6	\$3

The *Habeñero Loco* people decide to use **Knaster's Procedure** for the next meal order.

- Who gets to be next in line?

Katy

- How much does everyone get after Step 1?

Elizabeth gets 2, Cara gets 1, Katy pays 6 and gets to be next in line.

- How much does the winner pay in Step 1?

6

- How much surplus is there?

There is 6-2-1=3 dollars of surplus

Example 19 (Mean Girls 2). Three soon-to-be-former friends decide to never see one another again. Unfortunately they share a prized possession: **The Burn Book** . The chart below represents what they each would bid on this item:

	Regina	**Gretchen**	**Karen**
The Burn Book	$300	$210	$240

The Mean Girls decide to use **Knaster's Procedure**.

	Regina	Gretchen	Karen	
Fair Share	100	70	80	
Value of Item(s) Received	300	0	0	**Surplus ↓**
Pays/Gets	-200	70	80	-50

$X_{Regina} = 100 + \frac{50}{3} = 116.67$

$X_{Gretchen} = 70 + \frac{50}{3} = 86.67$

$X_{Karen} = 80 + \frac{50}{3} = 96.67$

Example 20 (Arrested Development 2). Lindsay and Tobias decide to divorce (again!).

	Lindsay	**Tobias**
Diamond Skin Cream	$200	$260
Poofy Blouse/Shirt	$100	$80

They decide to use **Knaster's Procedure**.

	Lindsay	Tobias	
Fair Share	150	170	
Value of Item(s) Received	100	260	**Surplus ↓**
Pays/Gets	50	-90	40
Compensation	170	190	

Each person recieves 20 of surplus.
Lindsay recieves the blouse and 70 dollars.
Tobias recieves the cream and pays 70 dollars

Example 21 (Miley and Liam 2). Miley and Liam have broken up and must divide some items they bought together. All amounts are thousands of dollars:

	Miley	**Liam**
Engagement Ring	\$400	\$250
Warehouse of Teddy Bear Costumes	\$100	\$60
Basic Human Dignity	\$30	\$200

They decide to use **Knaster's Procedure**.

	Miley	Liam	
Fair Share	265	255	
Value of Item(s) Received	500	200	**Surplus ↓**
Pays/Gets	-235	55	-180
Compensation	355	345	

Each person recieves 90 of surplus. Miley recieves the ring and warehouse and pays 145. Liam recieves the dignity and 145.

Example 22 (Practice: Knaster's Procedure). Three siblings inherit a car and a house from their parents. The siblings *HATE* each other and require mediation to determine who receives the items. They each make the following bids (in thousands of dollars):

	Sibling 1	Sibling 2	Sibling 3
House	300	600	450
Car	12	9	15

They decide to use **Knaster's Procedure**.

	Sibling 1	Sibling 2	Sibling 3
Fair Share	104	203	155
Value of Item(s) Received	0	600	15
Pays/Gets	104	-397	140
Compensation	155	254	206

surplus
-153

Each person recieves 51 of surplus

Compensation Method: Random

Definition 23 (Compensation Method: Random).

1. Each person places a bid on the disputed resource(s).
2. Some form of a **mediator** distributes the resources (and possibly money) amongst the people in a random way that need not be based upon the bids. This mediator could be an actual person, perhaps court-appointed, or some other random influence.

Example 24 (Compensation Method: Random). Two brothers, Sam and John, go to court to decide who will inherit their parents' car. Sam bids \$12 thousand for the car while John bids \$16 thousand for the car. In deciding who will get the car, the judge takes into consideration John's past issues with alcoholism and driving while drunk. The judge decides to award the car to Sam, even though he made a smaller bid.

Here is an Accounting for a Random Procedure:

	Person 1	Person 2	Person 3	...
Value of Item(s) Received		*Info you fill in*		
+ Amount of Cash Received		*Info you fill in*		
− Amount of Cash Paid		*Info you fill in*		
Compensation=				

An easy built-in check is this:

The **TOTAL** cash received *MUST* equal the **TOTAL** cash paid.

There is another built-in check related to compensation and the winning bid(s).

Example 25 (Caesar and Cleopatra 2). Pompey agrees to mediate in the dispute between Caesar and Cleopatra . Recall that Caesar thinks the The Port of Alexandria is worth 200 thousand denarii while Cleopatra believes it is worth only 160 thousand denarii.

- Suppose Pompey awards to port to Cleopatra but requires her to pay Caesar 100 thousand denarii.
  What is x_{Cleo} for this compensation arrangement?

  $X_{Cleo} = 160 - 100 = 60$

- Suppose Pompey awards to port to Caesar but negotiates a deal to pay Cleopatra 30 thousand denarii.
  What is x_{Caesar} for this compensation arrangement?

  $X_{Caesar} = 200 - 30 = 170$

Related Idea: Winning Bid Theorem

There is a connection between some of the pieces we have defined.

Theorem (Related Idea: Winning Bid Theorem). In any compensation scenario, the amount of the winning bid must equal the sum of the compensations.

Using our established notation, in a scenario with N people we have

$$w = x_1 + x_2 + x_3 + \ldots + x_N.$$

In particular, if we assume $w = b_1$ (So Person 1 is the winning bidder) then

$$x_1 = w - x_2 - x_3 - \ldots - x_N.$$

In words, the winning bidder pays everyone else with their winning bid for the right to keep a particular resource. The winning bidder's compensation is the value of the resource minus all other compensations to be paid.

Example 26 (Caesar and Cleopatra 3). Pompey agrees to mediate in the dispute between Caesar and Cleopatra . Recall that Caesar thinks the The Port of Alexandria is worth 200 thousand denarii while Cleopatra believes it is worth only 160 thousand denarii.

- Pompey awards to The Port of Alexandria to Cleopatra but requires her to pay Caesar 120 thousand denarii .

  What is x_{Cleo} for this compensation arrangement?

  $X_{Cleo} = W - X_{Caesar} = 160 - 120 = 40$

- Caesar and Cleopatra start arguing over the The Palace of Alexandria . Caesar thinks the The Palace of Alexandria is worth 40 thousand denarii while Cleopatra believes it is worth 160 thousand denarii. Pompey intervenes, awards the The Palace of Alexandria to Caesar but asks him to give Cleopatra 10 thousand denarii. What is the **TOTAL** compensation for Caesar and Cleopatra ?

  $X_{Caesar} = 40 - 10 = 30$

  $X_{Cleo} = 10$

Example 27 (Mean Girls 3). Three soon-to-be-former friends decide to never see one another again. Unfortunately they share a prized possession:

The Burn Book . The chart below represents what they each would bid on this item:

	Regina	Gretchen	Karen
The Burn Book	\$300	\$210	\$240

A mediator awards the The Burn Book to Karen who must pay Regina \$100 and Gretchen \$60.

- What is x_{Regina} for this arrangement?

$X_{Regina} = 100$

- What is $x_{Gretchen}$ for this arrangement?

$X_{Gretchen} = 60$

- What is x_{Karen} for this arrangement?

$X_{Karen} = 240 - 100 - 60 = 80$

Related Idea: Perception Chart

Definition 28 (Related Idea: Perception Chart). A person can determine how they view the everyone's compensation. CASH IS ALWAYS VIEWED AT FACE VALUE!

For Person k who receives a resource, Person i views them getting the value of the object in their eyes b_i, minus the value that Person k had to pay to receive the resource. The **Perception Chart** is made by finding all possible values.

Example 29 (Perception Chart).

Person	1	2	3 (Resource Winner)
Bids	b_1=0	b_2=10	b_3=30
Compensation	$x_1 = 5$	$x_2 = 15$	$x_3 = 30 - 15 - 5 = 10$

View of Compensation	1	2	3
1 views	5	15	-20
2 views	5	15	-10
3 views	5	15	10

Example 30 (Miley and Liam 3). Recall that Miley and Liam have broken up and the chart below represents what they each would bid on shared items (all in thousands of dollars):

	Miley	**Liam**
Engagement Ring	$400	$250
Warehouse of Teddy Bear Costumes	$100	$60
Basic Human Dignity	$30	$200

They decide to use **Knaster's Procedure** to divide the goods.

- Do Miley and Liam each get their fair share?

Yes.

- Make the Perception Chart for this arrangement:

Miley pays 145 and recieves the ring and warehouse. Liam recieves 145 and the dignity.

View of Compensation	Miley	Liam
Miley views	355	175
Liam views	165	345

Would Miley or Liam want to switch deals?

No, they both percieve themselves as getting a better deal than the other person.

Example 31 (Mean Girls 4). Three soon-to-be-former friends decide to never see one another again. Unfortunately they share a prized possession: **The Burn Book**. The chart below represents what they each would bid on this item:

	Regina	Gretchen	Karen
The Burn Book	$300	$210	$240

A mediator awards the The Burn Book to Karen who must pay Regina $100 andGretchen $60.

- Make the Perception Chart for this arrangement.

View of Compensation	Regina	Gretchen	Karen
Regina views	100	60	140
Gretchen views	100	60	50
Karen views	100	60	80

Example 32 (Arrested Development 3). Lindsay and Tobias decide to divorce (again!). Since they don't own much there aren't a lot of shared assets to fight over. However, there are two items that both people want; **Diamond Skin Cream** and a **Poofy Blouse/Shirt**. Here are their bids:

	Lindsay	Tobias
Diamond Skin Cream	$200	$260
Poofy Blouse/Shirt	$100	$80

A mediator decides that Tobias should get both items and pay Lindsay $120.

- Make the Perception Chart for this arrangement.

View of Compensation	Lindsay	Tobias
Lindsay views	120	180
Tobias views	120	220

Example 33 (Arrested Development 4). Lindsay and Tobias decide to divorce (again!). Since they don't own much there aren't a lot of shared assets to fight over. However, there are two items that both people want; **Diamond Skin Cream** and a **Poofy Blouse/Shirt**. Here are their bids:

	Lindsay	Tobias
Diamond Skin Cream	$200	$260
Poofy Blouse/Shirt	$100	$80

The mediator changes his mind and decides that Lindsay should get the **Poofy Blouse/Shirt** and Tobias should get the **Diamond Skin Cream** but pay Lindsay $40.

- Make the Perception Chart for this arrangement.

View of Compensation	Lindsay	Tobias
Lindsay views	140	160
Tobias views	120	220

Related Idea: The x/b Fractions

Definition 34 (Related Idea: The x/b Fractions)**.** In a compensation scenario, Person k places the bid b_k. Depending on the method used, Person k will also receive some value of compensation x_k. The fraction x_k/b_k gives a decimal that represents the overall value a person receives compared with the value they bid. **The higher the x/b fraction is for a person, the better they will feel about the deal they receive.**

Example 35 (The x/b Fractions)**.** Sam bids \$12 thousand for the car while John bids \$16 thousand for the car. A judge awards the car to Sam, but orders that Sam must pay John \$4 thousand dollars.

- John bid $b_{John} = 16$ and he receives $x_{John} = 4$ in compensation. So John's x/b fraction is $\frac{4}{16} = .25$.
- Sam bid $b_{Sam} = 12$ and he receives $x_{Sam} = 12 - 4 = 8$ in compensation. So Sam's x/b fraction is $\frac{8}{12} = .75$.

Example 36 (Mean Girls 5)**.** Three soon-to-be-former friends decide to never see one another again. Unfortunately they share a prized possession: **The Burn Book** . The chart below represents what they each would bid on this item:

	Regina	Gretchen	Karen
The Burn Book	\$300	\$210	\$240

A mediator awards the The Burn Book to Karen who must pay Regina \$100 and Gretchen \$60.

- What is the x/b fraction for Regina ?

$\frac{x}{b} = \frac{100}{300} = 0.33$

- What is the x/b fraction for Gretchen ?

$\frac{x}{b} = \frac{60}{210} = 0.29$

- What is the x/b fraction for Karen ?

$$\frac{x}{b} = \frac{80}{240} = 0.33$$

Example 37 (Arrested Development 5)**.** Lindsay and Tobias decide to divorce (again!). Since they don't own much there aren't a lot of shared assets to fight over. However, there are two items that both people want; **Diamond Skin Cream** and a **Poofy Blouse/Shirt**. Here are their bids:

	Lindsay	**Tobias**
Diamond Skin Cream	\$200	\$260
Poofy Blouse/Shirt	\$100	\$80

A mediator decides that Tobias should get both items and pay Lindsay \$120.

- What is the x/b fraction for Lindsay ?

$$\frac{x}{b} = \frac{120}{300} = 0.4$$

- What is the x/b fraction for Tobias ?

$$\frac{x}{b} = \frac{220}{340} = 0.65$$

- Who got the "better" deal?

Tobias since he has the higher x/b fraction.

Related Idea: Compensation Basics 4

Definition 38 (Related Idea: Compensation Basics 4)**.**

- We will often also need to know the **average bid** that people place, denoted by the symbol m (for *mean*, another word for average). This is just the total amount bid by everyone, divided by the number of people. In symbols this is

$$m = \frac{b_1 + b_2 + b_3 + \ldots + b_N}{N}.$$

Example 39 (Related Idea: Compensation Basics 4). Three sisters, April, May, and June, must decide who will inherit their parents' house. They make the following bids (in thousands of dollars):

$b_{April} = \$200 \qquad b_{May} = \$400 \qquad b_{June} = \$900$

The average bid is then

$$m = \frac{200 + 400 + 900}{3} = \$500(\text{thousand})$$

Example 40 (Miley and Liam 4). Miley and Liam have broken up and must divide some items they bought together. The chart below represents what they each would bid on these items (all in thousands of dollars):

	Miley	**Liam**
Engagement Ring	\$400	\$250
Warehouse of Teddy Bear Costumes	\$100	\$60
Basic Human Dignity	\$30	\$200

- What would m be for just the Engagement Ring ?

$$m = \frac{400+250}{2} = 325$$

- What would m be for just the Basic Human Dignity ?

$$m = \frac{30+200}{2} = 115$$

- What is m for the *ENTIRE* compensation scenario?

$$m = \frac{400+100+30+250+60+200}{2} = 520$$

Compensation Method: Equitability Procedure

Definition 41 (Compensation Method: Equitability Procedure). 1(a) Every person (all N of them) secretly bid on a resource.

1(b) The highest bidder receives the resource and pays $w - \frac{w}{N}$ to the mediator.

1(c) The mediator gives each non-winning person their fair share.

Notice that Step 1 (a)–(c) is the same as Knaster's Procedure and again may produce a **surplus**.

2 In addition, each person receives a proportion of the surplus.

First, some notation: $N \cdot m = b_1 + b_2 + b_3 + \ldots + b_N$.

- Person 1 receives: $\frac{b_1}{N \cdot m}(w - m)$.
- Person 2 receives: $\frac{b_2}{N \cdot m}(w - m)$.
- In general, Person k receives: $\frac{b_k}{N \cdot m}(w - m)$.

Example 42 (Time is Money 1). Two Professors bid to for a Campus Parking Spot:

	Math	**Biology**
Spot #19	\$60	\$80

The two Professors decide to use the **Equitability Procedure**. At the end of Step 1 of Knaster's Procedure, the Biology Professor gets the spot but pays \$40 and the Math Professor gets \$30. The surplus is \$10.

- What is m?

$m = \frac{60+80}{2} = 70$

- What is $N \cdot m$?

$N = 2$, $N \cdot m = 2 \cdot 70 = 140$

- Calculate $\frac{b_{Math}}{N \cdot m}(w - m)$. $W = 80$

$\frac{60}{140} \cdot (80 - 70) = 4.29$

- Calculate $\frac{b_{Biology}}{N \cdot m}(w - m)$.

$\frac{80}{140} \cdot 10 = 5.71$

- What is the compensation for each Professor?

$X_{Math} = 34.29$

$X_{Bio} = 45.71$

Beyond the Formula: Equitability Procedure

Let's go beyond the formula for the Equitability Procedure by looking at the components piece-by-piece. In general, the amount given to Person k in Step 2 is

$$\frac{b_k}{N \cdot m} \cdot (w - m)$$

You can work with our notation to note the following:

- $(w - m)$ is just the **available surplus** made from the winner(s) paying for the resource(s).
- $N \cdot m = b_1 + b_2 + b_3 + \ldots + b_N$ is the **sum of all bids**.
- We can describe the formula above using words as

$$\frac{\text{A Person's Bid(s)}}{\text{Sum of All Bids}} \cdot (\text{Available Surplus})$$

  This is an especially helpful view when there is **MORE THAN ONE RESOURCE!**

Example 43 (Mean Girls 6). Our Average Young Ladies from before decide to use the **Equitability Procedure** for disputed resource.

	Regina	**Gretchen**	**Karen**
The Burn Book	\$300	\$210	\$240

From Step 1 of Knaster's, Regina gets the book but pays \$200. Gretchen receives \$70 and Karen receives \$80. The surplus is \$50.

- What is $N \cdot m$? $m = \frac{300 + 210 + 240}{3} = 250$

$3 \cdot m = 3 \cdot 250 = 750$

- Calculate $\frac{b_{Regina}}{N \cdot m}(w - m)$.

$\frac{300}{750} \cdot 50 = 20$

- Calculate $\frac{b_{Gretchen}}{N \cdot m}(w - m)$.

$\frac{210}{750} \cdot 50 = 14$

- Calculate $\frac{b_{Karen}}{N \cdot m}(w - m)$.

$\frac{240}{750} \cdot 50 = 16$

Example 44 (Arrested Development 6). Recall that Lindsay and Tobias are fighting over **Diamond Skin Cream** and a **Poofy Blouse/Shirt**. Here are their bids:

	Lindsay	**Tobias**
Diamond Skin Cream	\$200	\$260
Poofy Blouse/Shirt	\$100	\$80

They decide to use the **Equitability Procedure** to divide the goods. Recall that after Step 1 of Knaster's Procedure, Tobias gets the Diamond Skin Creamand pays \$90 while Lindsay gets the Poofy Blouse/Shirtand receives \$50. The surplus is \$40.

- Calculate $\frac{b_{Tobias}}{N \cdot m}(w - m)$. $m = \frac{200+100+260+80}{2} = 320$, $w = 260+100 = 360$

$\frac{340}{640} \cdot 40 = 21.25$

- Calculate $\frac{b_{Lindsay}}{N \cdot m}(w - m)$.

$\frac{300}{640} \cdot 40 = 18.75$

- Find x_{Tobias} and $x_{Lindsay}$. What are $\frac{x_{Tobias}}{b_{Tobias}}$ and $\frac{x_{Lindsay}}{b_{Lindsay}}$?

$X_{Tobias} = 170 + 21.25 = 191.25$

$X_{Lindsay} = 150 + 18.75 = 168.75$

$\frac{X_T}{b_T} = \frac{191.25}{340} = 0.5625$

$\frac{X_L}{b_L} = \frac{168.75}{300} = 0.5625$

Example 45 (Miley and Liam 5, Part 1). Recall that Miley and Liam have broken up and the chart below represents what they each would bid on shared items (all in thousands of dollars):

	Miley	**Liam**
Engagement Ring	$400	$250
Warehouse of Teddy Bear Costumes	$100	$60
Basic Human Dignity	$30	$200

They decide to use the **Equitability Procedure** to divide the goods.

- Determine the TOTAL SURPLUS at the end of Step 1.

$w = 400 + 100 + 200 = 700, \; m = 520$

$\text{Surplus} = w - m = 700 - 520 = 180$

- Calculate $\frac{b_{Miley}}{N \cdot m}(w - m)$ and $\frac{b_{Liam}}{N \cdot m}(w - m)$.

$\frac{b_{Miley}}{N \cdot m}(w-m) = \frac{530}{1040} \cdot 180 = 91.73 \quad \frac{b_{Liam}}{N \cdot m}(w-m) = \frac{510}{1040} \cdot 180 = 88.27$

- Find x_{Miley} and x_{Liam}.

$X_{Miley} = 265 + 91.73 = 356.73$

$X_{Liam} = 255 + 88.27 = 343.27$

Example 46 (Miley and Liam 5, Part 2). Recall that Miley and Liam have broken up and the chart below represents what they each would bid on shared items (all in thousands of dollars):

	Miley	Liam
Engagement Ring	\$400	\$250
Warehouse of Teddy Bear Costumes	\$100	\$60
Basic Human Dignity	\$30	\$200

They decide to use the **Equitability Procedure**:

- Compare $\frac{x_{Miley}}{b_{Miley}}$ with $\frac{x_{Liam}}{b_{Liam}}$.

$\frac{X_M}{b_M} = 0.67 \quad \frac{X_L}{b_L} = 0.67$

- Suppose another compensation arrangement makes $x_{Miley} = \$450$ and $x_{Liam} = \$300$ (thousand) . Is this better or worse than the result of the Equitability Procedure?

$\frac{X_M}{b_M} = 0.85, \quad \frac{X_L}{b_L} = 0.59$

The x/b fractions are no longer equal!

Example 47 (Arrested Development 7, Part 1). Because Lindsay and Tobias have no money they decide to use a Point System for making bids. Each person gets 100 Points to distribute to desired items. Here are their bids:

	Lindsay	Tobias
Diamond Skin Cream	20	60
Poofy Blouse/Shirt	80	40

- Who should get the **Diamond Skin Cream**?

Tobias

- Who should get the **Poofy Blouse/Shirt**?

Lindsay

- If each person gets one item, is there any reason for someone to not like this compensation arrangement?

Tobias feels like he got less than Lindsay.

Comp. Method: Adjusted Winner for N=2

Definition 48 (Comp. Method: Adjusted Winner for N=2, Part 1). 1: Person 1 and Person 2 each have 100 points to use in making their bids. The first step is to assign points to the resources that are to be divided. The higher number of points given, the more a person wants that particular resource.

2: At first, the Highest Bidder for each resource gets that resource. For each item/resource received, the points used to get that resource must be Counted.

Tie Breaker: If there is a tie (same number of points) for a resource, it will go to the person who has used the fewest points so far.

The goal is for each person to receive resources *USING THE SAME NUMBER OF POINTS.*

The next step is to "Adjust" the winner of some resource(s) so that an equal number of points has been spent by both people.

Definition 49 (Comp. Method: Adjusted Winner for N=2, Part 2). **3:** If points used are not equal then resources are transferred from the person with the *HIGHER* number of points to the person with the *LOWER* number of points. Start by transferring items with the smallest **Point Ratio** (defined on the next slide).

4: If transferring an item moves too many points (causing the *HIGHER* and *LOWER* people to switch) then only a portion of the last item is transferred over. A **Transfer Equation** must be set up and solved to determine the amount shared. (Detailed descriptions of these equations are given later in the slides.)

The **Adjusted Winner Method** was first described in 1995.

Related Idea: Point Ratio

Definition 50 (Related Idea: Point Ratio)**.** For the Adjusted Winner Method, the **Point Ratio** for each resource is the fraction

$$\frac{\text{Points used by Winner of Resource}}{\text{Points used by Non-Winner of Resource}}.$$

Example 51 (Point Ratio)**.** Siblings GOB and Michael bid on the following:

	GOB	**Michael**	**Point Ratio**
Banana Stand	15	50	$\frac{50}{15} = 3.33$
Marta	60	10	$\frac{60}{10} = 6$
President's Office	25	40	$\frac{40}{25} = 1.6$
Points Used	60	90	

Related Idea: Transfer Equation

Definition 52 (Related Idea: Transfer Equation)**.** The **Transfer Equation** for a shared item is given by

$$\text{Person 1 non-shared points} + x \cdot \text{P1 Points of shared item} =$$
$$\text{Person 2 non-shared points} + (1 - x) \cdot \text{P2 Points of shared item.}$$

Here x represents the amount of the shared resource owned by P1.

Example 53 (Transfer Equation)**.** In the previous example, GOB will initially be awarded Marta while Michael will initially receive the Banana Stand and the President's Office. The President's Office has the lowest *Point Ratio*, so it must be shared. GOB has 60 (from Marta) non-shared points while Michael has 30 (from the Banana Stand). The transfer equation is

$$60 + x \cdot 25 = 50 + (1 - x) \cdot 40.$$

Example 54 (Arrested Development 7, Part 2)**.** Because Lindsay and Tobias have no money they decide to use a Point System for making bids. Each person gets 100 Points to distribute to desired items. Here are their bids:

	Lindsay	**Tobias**
Diamond Skin Cream	20	60
Poofy Blouse/Shirt	80	40

- What is the **Point Ratio** for each item?

D.S.C. : $\frac{60}{20} = 3$ Blouse : $\frac{80}{40} = 2$

- Who will get to keep 100% of an item (in other words, who will <u>NOT</u> have to transfer any ownership).

Tobias

- Solve the **Transfer Equation** below:

$$0 + x \cdot 80 = 60 + (1 - x) \cdot 40.$$

What does x mean here?

$x \cdot 80 = 60 + 40 - x \cdot 40$

$x 120 = 100$

$x = 0.83$

x is the amount of the Blouse Linsay keeps

Example 55 (Miley and Liam 6)**.** Miley and Liam decide to use the **Adjusted Winner Method**. Each person gets 100 Points to distribute to desired items. Here are their bids:

	Miley	Liam
Engagement Ring	50	20
Warehouse of Teddy Bear Costumes	20	10
Basic Human Dignity	30	70

- Who should get each item?

Miley should get the ring and warehouse
Liam should get the dignity.

- How many points does each person "spend" to get their items? 70
- Is there any reason for someone to not like the compensation arrangement at this stage?

No

Example 56 (Time is Money 2). Our Professors at the DMV decide they have even more to argue about! They decide to use the **Adjusted Winner Method** with bids below:

	Math	Biology
Parking Spot Near Math Build.	30	0
Parking Spot Near Rec. Center	20	10
Chair of University Salary Comm.	40	80
Spot #19 in line	10	10

pt ratio undef.
$20/10 = 2$
$80/40 = 2$
$10/10 = 1$

- Who should get each resource?

Math prof. gets the two parking spots and spot #19. Bio prof gets to chair the committee.

- How many points does the **Math Professor** use?

$30+20+10=60$

- How many points does the **Biology Professor** use?

80

- Find the **Point Ratio** for each item?

See above

- Which item will have (partial) ownership transferred?

Chair of the committee.

- Solve the **Transfer Equation** below:

$$0 + x \cdot 80 = 60 + (1 - x) \cdot 40.$$

Same equation as before, $x = 0.83$

What does x mean here?

X is the amount of the chair that the biology prof. keeps.

Example 57 (Keeping Up 1). Kris and Bruce are getting a divorce and the court orders that use the **Adjusted Winner Method** to divided their shared resources. They place the bids below:

	Kris	Bruce	pt ratio
Malibu House	60	50	60/50 = 1.2
Credit for Plastic Surgery	10	20	20/10 = 2
Country Club Membership	5	20	20/5 = 4
Louis Vuitton Luggage	25	10	25/10 = 2.5

shared item → Malibu House

- Who should get each resource?

  Kris: Malibu house, luggage
  Bruce - Credit, club memb.

- How many points does **Kris** use?

  85

- How many points does **Bruce** use?

  40

- What is the **Point Ratio** for each item?

  See above

- Set-up and solve the **Transfer Equation** for the example/numbers above.

  $25 + X \cdot 60 = 40 + (1-X) \cdot 50$

  $25 + X \cdot 60 = 40 + 50 - X \cdot 50$

  $X \cdot 60 + X \cdot 50 = 90 - 25$

  $\to X \cdot 110 = 65$

  $X = 0.59$

Example 58 (Caesar and Cleopatra 4). Caesar and Cleopatra are fighting over even more stuff! Answer the following questions about the **Adjusted Winner Method** with the bids below:

	Caesar	**Cleopatra**	
The Port of Alexandria	30	20	30/20 = 1.5
The Palace of Alexandria	20	20	20/20 = 1
Exotic Poison Collection	5	20	20/5 = 4
Legion of Soldiers	45	40	45/40 = 1.125

- Who gets each resource?

  Caesar : Port, Soldiers

  Cleo - Palace, poison

- Find the **Point Ratio** for each resource and determine which item must be shared.

  See above

  Soldiers are shared.

- Set up and solve the **Transfer Equation**.

  $30 + x \cdot 45 = 40 + (1-x)40$

  $30 + x \cdot 45 = 40 + 40 - x \cdot 40$

  $x \cdot 45 + x \cdot 40 = 80 - 30$

  $x \cdot 85 = 50$

  $x = 0.59$

Good Compensation Result: Fair

Definition 59 (Good Compensation Result: Fair). A compensation arrangement is **Fair** whenever every person gets their fair share. If the scenario has N people then this means

Recall our previous notation of b_i and x_i for the bid and compensation of the ith Person. Then in symbols, **Fair** means

$$x_1 \geq \frac{b_1}{N}, \quad x_2 \geq \frac{b_2}{N}, \quad x_3 \geq \frac{b_3}{N}, \dots \quad x_N \geq \frac{b_N}{N}.$$

Example 60 (Fair). Suppose Larry, Curly, and Moe make bids on a cream pie. Larry bids $b_1 = \$12$, Curly bids $b_2 = \$9$ and Moe bids $b_3 = \$15$. In order for any compensation arrangement to be fair Larry must receive at least \$4 in compensation, Curly must receive at least \$3 in compensation, and Moe must receive at least \$5 in compensation.

Example 61 (Caesar and Cleopatra 5). Recall that Caesar and Cleopatra are fighting over The Port of Alexandria . Caesar thinks The Port of Alexandria is worth 200 thousand denarii. Cleopatra knows the port didn't cost that much to build and could easily be rebuilt for only 160 thousand denarii.

	Caesar	**Cleopatra**
The Port of Alexandria	200	160

- Suppose Caesar gets the Port and pays Cleopatra her fair share. Is this compensation arrangement fair?

Yes. $X_{Cleo} = \frac{160}{2}$ and $X_{Caesar} = 120 \geq 100$

- Suppose Cleopatra gets the Port and pays Caesar his fair share. Is this compensation arrangement fair?

No, $X_{Cleo} = 60 < 80 = \frac{160}{2}$

Big Idea: Fair Compensation Theorem

Theorem (Fair Compensation Theorem). A compensation arrangement can be fair only when the winning bid is greater than the average bid. Using our established notation, this means for an arrangement to be fair we must have

$$w \geq m.$$

Example 62 (Fair Compensation Theorem). Five siblings place bids (in thousands of dollars) $b_1 = 210,\ b_2 = 240,\ b_3 = 120,\ b_4 = 300,\ b_5 = 280$ on a house. A mediator will award the house to one of the siblings, but not necessarily the highest bidder. Since the average bid is

$$m = \frac{210 + 240 + 120 + 300 + 280}{5} = 230$$

the only ways to make a fair compensation arrangement is for Siblings 2, 4, or 5 to get the house.

Example 63 (Crazy Good Tacos 3). Recall the bids to be next in line at the *Habeñero Loco* taco truck:

	Katy	**Elizabeth**	**Cara**
Next in Line	\$9	\$6	\$3

- What is m for this scenario?

  $m = 6$

- Who could be next in line and still have a fair compensation arrangement?

  Katy and Elizabeth.

Example 64 (Sibling Rivalry 1). Four siblings make a claim on the "Really Cool Jacket". A judge orders them to submit secret bids for the Really Cool Jacket. The siblings give the following bids (in millions of dollars):

	Larry	**Harry**	**Jerry**	**Mary**
Really Cool Jacket	\$124	\$140	\$116	\$164

- What is m for this scenario?

  $m = 136$

- Which Sibling(s) could receive the "Really Cool Jacket" and still have a fair compensation arrangement?

  Harry and Mary

Example 65 (Sibling Rivalry 2). Four siblings make a claim on the "Really Cool Jacket". A judge orders them to submit secret bids for the Really Cool Jacket. The siblings give the following bids (in millions of dollars):

	Larry	**Harry**	**Jerry**	**Mary**
Really Cool Jacket	\$124	\$140	\$116	\$164

The judge awards the "Really Cool Jacket" to the Harry who agrees pay everyone \$36.

- Find the x/b fractions for all siblings.

$$\frac{x_L}{b_L} = \frac{36}{124} = 0.29, \quad \frac{x_H}{b_H} = \frac{32}{140} = 0.23$$

$$\frac{x_J}{b_J} = \frac{36}{116} = 0.31, \quad \frac{x_M}{b_M} = \frac{36}{164} = 0.22$$

- Using your answer above, which sibling gets the "best" deal?

Jerry

Good Compensation Result: Equitable

Definition 66 (Good Compensation Result: Equitable). A compensation arrangement is **Equitable** if each person answers the question "What fraction of the total value did you get?" in *EXACTLY THE SAME WAY.*

Using our notation, if there are N people in the scenario then we must have

$$\frac{x_1}{b_1} = \frac{x_2}{b_2} = \frac{x_3}{b_3} = \cdots = \frac{x_N}{b_N}$$

in order for the compensation to be **Equitable**. In other words,
EVERYONE HAS THE SAME x/b FRACTION!

Example 67 (Equitable). Suppose Jack and Jill now place bids of $b_{Jack} = \$14$ and $b_{Jill} = \$6$ on magic beans. Jack gets the beans but pays Jill $x_{Jill} = \$4.20$.

$$\frac{x_1}{b_1} = \frac{4.2}{6} = 0.7 \qquad \frac{x_2}{b_2} = \frac{9.8}{14} = 0.7$$

Example 68 (Caesar and Cleopatra 6). Recall that Caesar and Cleopatra are fighting over The Port of Alexandria . Caesar thinks The Port of

Alexandria is worth 200 thousand denarii. Cleopatra knows the port didn't cost that much to build and could easily be rebuilt for only 160 thousand denarii.

	Caesar	**Cleopatra**
The Port of Alexandria	200	160

- Suppose Caesar gets the The Port of Alexandria and pays Cleopatra 90 thousand denarii. Is this compensation arrangement equitable?

No. $\frac{x_{cleo}}{b_{cleo}} = 0.5625$ and $\frac{x_{cae}}{b_{cae}} = \frac{110}{200} = 0.55$

- Suppose Caesar gets the The Port of Alexandria and pays Cleopatra 88.89 thousand denarii. Is this compensation arrangement equitable?

Yes $\frac{x_{cleo}}{b_{cleo}} = \frac{88.89}{160} = 0.556$ and $\frac{x_{cae}}{b_{cae}} = \frac{111.11}{200} = 0.556$

Example 69 (Mean Girls 7). Our Mean Girls bid on **The Burn Book** , but at the last minute **Karen** increases her bid:

	Regina	**Gretchen**	**Karen**
The Burn Book	\$300	\$210	\$270

A mediator awards the book to **Karen** , who must then pay **Regina** and **Gretchen** their fair share.

- What are x_{Regina}, $x_{Gretchen}$, and x_{Karen}?

$x_R = 100$, $x_G = 70$, $x_K = 100$

- Calculate the x/b fractions for Regina and Gretchen .

$\frac{x_R}{b_R} = \frac{100}{300} = 0.33$, $\frac{x_G}{b_G} = \frac{70}{210} = 0.33$

- Is this compensation arrangement equitable?

No, since $\frac{x_K}{b_K} = \frac{100}{270} = 0.37$

Example 70 (Arrested Development 8). Recall that Lindsay and Tobias are fighting over **Diamond Skin Cream** and a **Poofy Blouse/Shirt**. Here are their bids:

	Lindsay	**Tobias**
Diamond Skin Cream	\$200	\$260
Poofy Blouse/Shirt	\$100	\$80

They decide to use the **Knaster's Procedure** to divide the goods. Recall that after Knaster's Procedure, Tobias gets the Skin Cream and pays \$70 while Lindsay gets the Blouse and receives \$70.

- What is $x_{Lindsay}/b_{Lindsay}$?

  $\frac{X_L}{b_L} = \frac{170}{300} = 0.57$

- What is x_{Tobias}/b_{Tobias}?

  $\frac{X_T}{b_T} = \frac{190}{340} = 0.56$

- Is this compensation arrangement equitable?

  No

- Do you think Knaster's Procedure will produce a equitable compensation arrangement? Why or why not?

  No

Example 71 (Miley and Liam 7). Recall the bids from Miley and Liam (in thousands of dollars):

	Miley	Liam
Engagement Ring	\$400	\$250
Warehouse of Teddy Bear Costumes	\$100	\$60
Basic Human Dignity	\$30	\$200

After using the **Equitability Procedure**, Miley gets the Engagement Ring and Warehouse of Teddy Bear Costumes , but pays Liam \$143.27 (thousand). Liam gets Basic Human Dignity and receives \$143.27 (thousand).

- What are x_{Miley}/b_{Miley} and x_{Liam}/b_{Liam}?
- Is this compensation arrangement equitable?

$\frac{X_M}{b_M} = \frac{356.73}{530} = 0.67$, $\frac{X_L}{b_L} = \frac{343.27}{510} = 0.67$

Yes

- Do you think Equitability Procedure will produce an equitable compensation arrangement? Why or why not?

Yes

Related Idea: Finding Equitable Amounts for $N = 2$

Let's assume that $b_2 = w = h$, so Person 2 is the winning/highest bidder. This means Person 2 keeps the resource but pays Person 1 to do so. The overall compensation for Person 2 is then the value of the resource minus the compensation paid to Person 1. In symbols this means $x_2 = b_2 - x_1$. For the compensation to be equitable, we need

$$\frac{b_1}{x_1} = \frac{b_2}{b_2 - x_1}.$$

This leads to following shortcut.

Theorem (Finding Equitable Amounts for $N = 2$). For $N = 2$, suppose Person 2 is the winning/highest bidder. To compensate an equitable amount Person 2 should pay

$$x_1 = \frac{b_1 \cdot b_2}{b_1 + b_2}$$

to Person 1.

Good Compensation Result: Envy-Free

Definition 72 (Good Compensation Result: Envy-Free)**.** Envy in a compensation arrangement occurs when one person views another person's compensation as more valuable than their own.
A compensation arrangement is **Envy-Free** (or **E-F** for short) if there is no envy. In other words, envy-free means each person values their compensation more (or possibly equal) compared to the compensation of any other person.

Example 73 (Envy-Free)**.** Consider part of the Perception Chart from a previous example.

View of Compensation	**1**	**2**	**3**
1 views	5	15	-20

Note that Person 1 views their compensation as *LESS* than the compensation for Person 2. In other words, Person 1 will *ENVY* the deal that Person 2 received. This says the corresponding compensation arrangement is <u>not</u> Envy-Free.

Example 74 (Mean Girls 8)**.** The chart below represents what each of would bid on this item:

	Regina	**Gretchen**	**Karen**
The Burn Book	$300	$210	$240

A mediator awards the The Burn Book to Karen who must pay Regina $100 and Gretchen $60.

Recall the Perception Chart for this arrangement.

View of Compensation	**Regina**	**Gretchen**	**Karen**
Regina views	100	60	140
Gretchen views	100	60	50
Karen views	100	60	80

- Would Karen like to trade compensation with anyone?

Karen would want to trade with Regina

- Would Regina like to trade compensation with anyone?

Regina would want to trade with Karen

Example 75 (Mean Girls 9). Our Mean Girls bid on **The Burn Book** , but at the last minute **Karen** increases her bid:

	Regina	**Gretchen**	**Karen**
The Burn Book	\$300	\$210	\$270

A mediator awards the book to **Regina** , who must then pay **Gretchen** and **Karen** their fair share.

- Make the Perception Chart for this compensation arrangement:

View of Compensation	**Regina**	**Gretchen**	**Karen**
Regina views	140	70	90
Gretchen views	50	70	90
Karen views	110	70	90

- Is this compensation arrangement Envy-Free?

No, for instance Gretchen envies Karen.

Big Idea: Envy-Free Compensation Theorem

Theorem (Envy-Free Compensation Theorem). For a compensation arrangement to be **Envy-Free (E-F)** the winning bidder must be the highest bidder *AND* all others receive equal compensation amounts not above the fair share of the winning bidder, but not below the fair share of any of the others.

In symbols, for **E-F** with N people we need:

- $w = h$;
- If Person 1 is the winning bidder, then $x_2 = x_3 = \ldots = x_N$;
- If Person 1 is the winning bidder, then $x_i \leq \frac{b_1}{N}$ for $i = 2, 3, \ldots, N$;
- $x_i \geq \frac{b_i}{N}$ for $i = 2, 3, \ldots, N$.

This theorem gives us a shortcut for determining if a compensation arrangement is envy-free. It is very useful!

Related Idea: Envy-Free For $N = 2$

Let's check the Envy-Free Theorem for $N = 2$ when it is easy to write down all of the pieces:

- $w = h$;
- If Person 1 is the winning bidder, then $x_2 = x_3 = \ldots = x_N$;
- If Person 1 is the winning bidder, then $x_i \leq \frac{b_1}{N}$ for $i = 2, 3, \ldots, N$;
- $x_i \geq \frac{b_i}{N}$ for $i = 1, 2, 3, \ldots, N$.

Assume that $w = b_1$. For the scenario to be E-F the winning bid must be the highest bid so $b_1 = h$. Then it follows that $b_1 \geq b_2$.

Note b_2 is equal to itself and there are no other equalities to check. Finally, for E-F we must have

$$\frac{b_2}{2} \leq x_2 \leq \frac{b_1}{2}.$$

Related Idea: Envy-Free=Fair For $N = 2$

From the previous section we have the following shortcut:

Theorem (Envy-Free=Fair for $N = 2$). For $N = 2$, if a compensation arrangement is Envy-Free then it is automatically Fair as well.

Likewise, for $N = 2$ if the compensation arrangement is Fair then it is also Envy-Free.

Example 76 (Envy-Free=Fair For $N = 2$). Jack and Jill each place bids of $b_{Jack} = \$10$ and $b_{Jill} = \$4$ on a bag of magic beans. Jack is awarded the beans and pays Jill $x_{Jill} = \$1$. Is this compensation arrangement envy-free?

Since Jill's Fair Share is $\frac{4}{2} = \$2$ this arrangement is not fair. By the above theorem, we know that this arrangement is also not envy-free. In fact, Jill knows she gets \$1 but she thinks Jack's compensation is worth \$4-\$1=\$3.

Example 77 (Sibling Rivalry 3). Four siblings make a claim on the "Really Cool Jacket". A judge orders them to submit secret bids for the Really Cool Jacket. The siblings give the following bids (in millions of dollars):

	Larry	**Harry**	**Jerry**	**Mary**
Really Cool Jacket	\$124	\$140	\$116	\$164

- Who should receive the collection if the arrangement is to be envy-free? Mary

- The winning Sibling pays the other Siblings \$28 each. Is this compensation arrangement envy-free?

No, Larry does not recieve his fair share.

- The winning Sibling pays the other Siblings \$40 each. Is this compensation arrangement envy-free?

Yes, all of the conditions of the theorem hold.

Example 78 (Mean Girls 10)**.** Our Average Young Ladies bid on **The Burn Book** , but at the last minute **Karen** increases her bid:

	Regina	**Gretchen**	**Karen**
The Burn Book	\$300	\$210	\$270

- Who should receive the book if the arrangement is to be Envy-Free?

Regina

- The winner pays everyone else \$80 each. Is this arrangement Envy-Free?

No, Karen does not get her fair share

- What is the minimum amount that the winner could pay for an Envy-Free arrangement?

90 dollars to each person

- What is the maximum amount that the winner could pay for an Envy-Free arrangement?

100 dollars to each person.

Example 79 (Caesar and Cleopatra 6). Recall that Caesar and Cleopatra are fighting over the The Port of Alexandria and The Palace of Alexandria . They make the following bids:

	Caesar	**Cleopatra**
Port of Alexandria	200	160
Palace	40	80

- Who should get what if the compensation arrangement is to be Envy-Free?

  Caesar should get the port, Cleo should get the palace.

- Suppose Caesar gets the Port and pays Cleopatra 90 thousand denarii. Is this compensation arrangement Envy-Free?

  No, $X_{Caesar} = 110 < \frac{b_{Caesar}}{2} = 120$

- Suppose Caesar gets the Port and pays Cleopatra 130 thousand denarii. Is this compensation arrangement Envy-Free?

  No for the same reason as above

Example 80 (Mean Girls 11). Our Average Young Ladies bid on **The Burn Book** , but at the last minute **Karen** increases her bid:

	Regina	**Gretchen**	**Karen**
The Burn Book	\$300	\$210	\$270

- Suppose a mediator awards the book to **Gretchen** , who must then pay **Regina** and **Karen** their fair share. Find x_{Regina}, $x_{Gretchen}$, and x_{Karen}.

  $X_R = 100, X_G = 210 - 190 = 20, X_K = 90$

- Suppose a mediator awards the book to **Karen** , who must then pay **Regina** and **Gretchen** their fair share. Find x_{Regina}, $x_{Gretchen}$, and x_{Karen}. $X_R = 100$, $X_G = 70$, $X_K = 100$

- Compare the two compensation scenarios above. Is one of them "better" than the other? Why or why not? The second one. all three come away with more.

Good Compensation Result: Paréto Optimal (P-O)

Definition 81 (Good Compensation Result: Pareto-Optimal (P-O)). A compensation arrangement is **Pareto-Optimal (P-O)** if there are no other arrangements that ALL people would prefer.

Suppose N people have received compensations $x_1, x_2, x_3, \ldots, x_N$. Before these compensations are given out someone suggests alternate compensations $\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots \hat{x}_N$.

Suppose in the second compensation arrangement Person 1 actually gets a better compensation. In symbols this says $x_1 \leq \hat{x}_1$. The first compensation arrangement is **Pareto-Optimal** if this forces have $x_k \geq \hat{x}_k$ for some k.

In words, a Pareto-Optimal compensation arrangement is one in which any improvement in compensation for one person comes at the expense of the other people involved.

Pareto Optimal is also sometimes called **Pareto-Efficient**.

While P-O is easily seen to be desirable, right now it is unclear how to tell if a compensation arrangement is P-O.

Example 82 (Arrested Development 9). Recall that Lindsay and Tobias are fighting over **Diamond Skin Cream** and a **Poofy Blouse/Shirt**. Here are their bids:

	Lindsay	**Tobias**
Diamond Skin Cream	\$200	\$260
Poofy Blouse/Shirt	\$100	\$80

- Suppose Lindsay gets the Diamond Skin Creambut pays Tobias his fair share for this item. Tobias gets the Poofy Blouse/Shirtbut pays Lindsay her fair share for this item. Find $x_{Lindsay}$ and x_{Tobias}.

$X_L = 200 - 130 + 50 = 120$

$X_T = 80 - 50 + 130 = 160$

- Now suppose Tobias and Lindsay switch items with Lindsay paying Tobias his fair share for the Poofy Blouse/Shirtand Tobias paying Lindsay her fair share for the Diamond Skin Cream. Find $x_{Lindsay}$ and x_{Tobias} ,

$X_L = 100 - 40 + 100 = 160$
$X_T = 260 - 100 + 40 = 200$

Big Idea: Pareto-Optimal Theorem

Theorem (Pareto-Optimal Theorem). A compensation arrangement is Pareto-optimal exactly when the winning bidder is the highest bidder. Using our established notation, this means for an arrangement to be Pareto-Optimal we must have

$$w = h.$$

Example 83 (Pareto-Optimal Theorem). Five siblings place bids (in thousands of dollars) $b_1 = 210,\ b_2 = 240,\ b_3 = 120,\ b_4 = 300,\ b_5 = 280$ on a house. A mediator decides to award the house to Sibling 2. Is there an argument against this arrangement?
The only way to make an arrangement Pareto-Optimal is for Sibling 4 (the highest bidder) to get the house. Any other sibling getting the house produces an arrangement where someone receives less in compensation.

Example 84 (Crazy Good Tacos 4). Recall the bids to be next in line at the *Habeñero Loco* taco truck:

	Katy	**Elizabeth**	**Cara**
Next in Line	\$9	\$6	\$3

- Suppose Cara goes next in line and pays everyone their fair share. Do you think this arrangement is Pareto-Optimal? Why or why not? No
- Suppose Katy goes next in line and pays everyone their fair share. Do you think this arrangement is Pareto-Optimal? Why or why not? Yes
- Suppose Elizabeth goes next in line and pays everyone their fair share. Do you think this arrangement is Pareto-Optimal? Why or why not? No

Example 85 (Sibling Rivalry 4). Four siblings make a claim on the "Really Cool Jacket". A judge orders them to submit secret bids for the Really Cool Jacket. The siblings give the following bids (in millions of dollars):

	Larry	**Harry**	**Jerry**	**Mary**
Really Cool Jacket	$124	$140	$116	$164

- Suppose Harry gets the collection and pays everyone else $48. Is this compensation arrangement Pareto-Optimal?

  No

- Who should receive the collection if the arrangement is to be Pareto-Optimal?

  Mary

Example 86 (Caesar and Cleopatra (And More) 7). More people demand to be part of Caesar and Cleopatra's deal. Below are bids for various people on a number of items:

	Caesar	**Cleo**	**Pompey**	**Octavian**	**Agrippa**
Port	200	160	180	220	100
Palace	40	80	60	30	90
Roman Legion	35	5	25	30	20

- Who should get the Port if the compensation arrangement is to be Pareto-Optimal?

  Octavian

- Who should get the Palace if the compensation arrangement is to be Pareto-Optimal?

  Agrippa

- Who should get the Roman Legion if the compensation arrangement is to be Pareto-Optimal?

Caesar

Related Idea: Compensation Summary

This is what we can conclude about compensation:

	Fair	**E-F**	**P-O**	**Equit.**	**Requires**
Knaster's Procedure	✓	X	✓	X	Money
Random	X	X	X	X	Money
Equitability Procedure	✓	X	✓	✓	Money
Adjusted Winner Method For $N = 2$	✓	✓	✓	✓	No Money

Remember that for $N = 2$, "FAIR=ENVY FREE"! So actually, for just two people **Knaster's Method** and the **Equitability Procedure** are better than they are in general.

What makes the **Adjusted Winner Method For Two People** better is that it DOES NOT require any money as compensation. This makes it a realistic option to use.

The **Adjusted Winner Method For Two People** has been called "Mathematics' Contribution to World Peace".